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# Note

# Using Partitions to Characterize the Minimum Cardinality of an Unbounded Family in ${}^{\omega}\omega$*

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Let $b$ be the minimum cardinality of an unbounded family in ${}^{\omega}\omega$ partially ordered by $\leqslant^*$. Recall that $f \leqslant^* g$ if $f(n) \leqslant g(n)$ for all but a finite number of $n$. We first generalize $\leqslant^*$ to ${}^{\nu}\omega$ for each infinite cardinal $\nu$. We then prove that $b > \nu$ iff every sequence in ${}^{\nu}\omega$ has a $\leqslant^*$-upper bound; we also show that the existence of an upper bound for a sequence $\sigma$, where $\sigma_k : \nu \to \omega$ for each $k \in \omega$, depends only on the partitions of $\nu$ induced by the point-inverse sets of the individual functions.


## INTRODUCTION AND THEOREM

*Partial orders.* For $f, g \in {}^{\omega}\omega$, define $f \leqslant^* g$ if $(\exists m \in \omega)\ (\forall n \geqslant m)\ (f(n) \leqslant g(n))$, i.e., if $f$ is eventually less than $g$.

By applying the eventual condition to the range instead of the domain, we can define a $\leqslant^*$-partial order on ${}^{\nu}\omega$, where $\nu$ is any infinite cardinal: $\sigma \leqslant^* \tau$ if $(\exists j \in \omega)\ (\forall i \geqslant j)\ (\forall \alpha \in \sigma^{-1}(i))\ (\sigma(\alpha) \leqslant \tau(\alpha))$. (In this paper we only consider functions in ${}^{\nu}\omega$ that assume an infinite number of values.) For $\nu = \omega$, we use the domain version of $\leqslant^*$ given in the preceding paragraph; it should be noted, however, that the domain and range versions are in agreement for increasing functions.

*Cardinal numbers.* With respect to ${}^{\omega}\omega$ partially ordered by $\leqslant^*$, let $b$ be the minimum cardinality of an unbounded family, and let $d$ be the minimum cardinality of a dominant (cofinal) family. These two cardinal

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numbers have received much attention in recent years. First note that in ZFC, $\omega_1 \leqslant b \leqslant d \leqslant c$ ($c$ is the cardinality of the reals). It is consistent with ZFC to simultaneously change any of these relations to either strict equality or strict inequality. In particular, both the continuum hypothesis and Martin's axiom imply $b=d=c$ (the equality $b=d$ holds iff there is a well-ordered dominant family (scale)). More generally, S. H. Hechler has shown that if $M$ is a model of ZFC in which $\gamma$ and $\delta$ are any two regular cardinals with $\omega_1 \leqslant \gamma \leqslant \delta \leqslant c$, then there is an extension $N$ of $M$ in which $b=\gamma$, $d=\delta$, and the aleph value of $c$ is unchanged [H].

*Subordination.* Suppose $v$ is an infinite cardinal. We use the term "partition" to refer to a partition of $v$ with a countably infinite number of cells. (If the image of a sequence is a partition, then the sequence itself will also be referred to as a partition.)

Suppose $\{\mu(n)\colon n \in \omega\}$ and $\{\mu'(n)\colon n \in \omega\}$ are partitions. Then $\mu$ is subordinate to $\mu'$ if each $\mu(n)$ is covered by a finite subcollection of $\{\mu'(n)\colon n \in \omega\}$. Equivalently, if a sequence in $v$ runs through $\mu'$, then it runs through $\mu$ also (where a sequence runs through a partition if each partition set contains at most a finite number of the terms).

THEOREM. *Suppose $v$ is a cardinal with $v \geqslant \omega_1$. Then $b > v$ iff every sequence in ${}^v\omega$ has an upper bound; in turn, a sequence $\langle \sigma_k \rangle$ in ${}^v\omega$ has an upper bound iff there is a partition $\mu$ such that for each $k \in \omega$, $\mu$ is subordinate to $\{\sigma_k^{-1}(i)\colon i \in \omega\}$.*

*Interchanging function and variable.* For each $\sigma \in {}^\omega({}^v\omega)$, let $\Phi_\sigma \in {}^v({}^\omega\omega)$, where for each $\alpha \in v$ we define $\Phi_{\sigma,\alpha}\colon \omega \to \omega$ by $\Phi_{\sigma,\alpha}(k) = \sigma_k(\alpha)$.

*Proof of the Theorem.* The transformation $\sigma \to \Phi_\sigma$, defined by interchanging function and variable, is a 1–1 correspondence from ${}^\omega({}^v\omega)$ onto ${}^v({}^\omega\omega)$. We will (implicitly) restrict $\Phi$ to sequences $\sigma$ where Image($\sigma_k$) is infinite for each $k$. By Lemmas 1 and 2 below, this restriction in either direction preserves boundedness: $\{\sigma_k : k \in \omega\} \subseteq {}^v\omega$ has an upper bound iff $\{\Phi_{\sigma,\alpha} : \alpha \in v\} \subseteq {}^\omega\omega$ has an upper bound; and the existence of an upper bound for either collection depends only on the partitions of $v$ induced by the point-inverse sets.

## LEMMAS

*Diagonal upper bounds.* Suppose that for each $k \in \omega$, $\sigma_k \in {}^v\omega$; and suppose there exists a partition $\mu$ such that for each $k$, $\mu$ is subordinate to $\{\sigma_k^{-1}(i)\colon i \in \omega\}$.

We define below $g = g_{\sigma,\mu} \in {}^\omega\omega$ and $\tau = \tau_{\sigma,\mu} \in {}^v\omega$. In the proof of Lemma 1 we show that $\tau$ is an upper bound for $\{\sigma_k : k \in \omega\}$, and in the proof of Lemma 2 we show that $g$ is an upper bound for $\{\Phi_{\sigma,\alpha} : \alpha \in \nu\}$.

Define a sequence of integer sets $I = I_{\sigma,\mu}$ by $I(n) = \{\sigma_k(\alpha)$: $k \leqslant n$ and $\alpha \in \mu(m)$ for some $m \leqslant n\}$. By subordination each of these integer sets has a largest element. Let $g$: $\omega \to \omega$ defined by $g(n) = \max I(n)$, and let $\tau$: $\nu \to \omega$ defined by $\tau(\alpha) = \max I(n)$, where $\alpha \in \mu(n)$.

To understand $I(n)$, first build a matrix of subsets of $\nu$ where the $k$th column holds in order the point-inverses of $\sigma_k$. Fix $n$. For each $k \leqslant n$, find the smallest initial segment of the $k$th column that covers $\bigcup_{m \leqslant n} \mu(m)$; then $\max I(n)$ is an upper bound for the image of $\sigma_k$ restricted to the union of the initial segment. With respect to this matrix, $g$ and $\tau$ are constructed by a diagonal procedure.

LEMMA 1. *Suppose that for each $k \in \omega$, $\sigma_k \in {}^{\nu}\omega$. Then $\{\sigma_k : k \in \omega\}$ has an upper bound in ${}^{\nu}\omega$ iff there exists a partition $\mu$ such that for each $k$, $\mu$ is subordinate to $\{\sigma_k^{-1}(i)$: $i \in \omega\}$.*

*Proof.* Necessity. Claim. Suppose $\xi$, $\eta \in {}^{\nu}\omega$ with $\xi \leqslant^* \eta$. Then the $\eta$-partition is subordinate to the $\xi$-partition. We first observe that a sequence $(\alpha_n)$ runs through the $\xi$-partition ($\eta$-partition) iff $\xi(\alpha_n) \to \infty$ ($\eta(\alpha_n) \to \infty$) as $n \to \infty$. Let $j \in \omega$ such that for every $i \geqslant j$ and for every $\alpha \in \xi^{-1}(i)$, $\xi(\alpha) \leqslant \eta(\alpha)$. Suppose $(\alpha_n)$ is a sequence in $\nu$ running through the $\xi$-partition. Then we can choose $m$ where $\xi(\alpha_n) \geqslant j$ for each $n \geqslant m$. Therefore by the choice of $j$, $\xi(\alpha_n) \leqslant \eta(\alpha_n)$ for each $n \geqslant m$. So $(\alpha_n)$ runs through the $\eta$-partition also.

Sufficiency. Claim. Suppose we have a subordinate partition $\mu$. Then $\tau = \tau_{\sigma,\mu}$ is an upper bound for $\{\sigma_k : k \in \omega\}$. Fix $k$, let $i \in \text{Image}(\sigma_k)$ with $i \geqslant \max I(k) + 1$, and let $\alpha \in \sigma_k^{-1}(i)$. Then by the definition of $I$, $\alpha \notin \mu(m)$ for each $m \leqslant k$. Let $n \in \omega$ with $\alpha \in \mu(n)$. By the preceding observation, $k < n$, so again using the definition of $I$, $\sigma_k(\alpha) \in I(n)$. We now have that $\tau(\alpha)$ is the largest integer in a set of integers that includes $\sigma_k(\alpha)$. So for every $i \geqslant \max I(k) + 1$ and for every $\alpha \in \sigma_k^{-1}(i)$, $\sigma_k(\alpha) \leqslant \tau(\alpha)$.

LEMMA 2. *Suppose that for each $k \in \omega$, $\sigma_k \in {}^{\nu}\omega$. Then $\{\Phi_{\sigma,\alpha} : \alpha \in \nu\}$ has an upper bound in ${}^{\omega}\omega$ iff there exists a partition $\mu$ such that for each $k$, $\mu$ is subordinate to $\{\sigma_k^{-1}(i)$: $i \in \omega\}$.*

*Proof.* Necessity. Suppose $h \in {}^{\omega}\omega$ such that $\Phi_{\sigma,\alpha} \leqslant^* h$ for each $\alpha \in \nu$. For each $k$ and each $n$, let $\Gamma_{k,n} = \bigcup \{\sigma_k^{-1}(i)$: $i \leqslant h(k) + n\}$. For each $n$, let $\Delta_n = \bigcap_{k \in \omega} \Gamma_{k,n}$, and let $\mu(n) = \Delta_n - \Delta_{n-1}$. An equivalent definition of $\Delta$ (due to the referee), is $\Delta_n = \{\alpha \in \nu$: $(\forall k \in \omega)$ $(\Phi_{\sigma,\alpha}(k) \leqslant h(k) + n)\}$.

Claim 1. The collection $\{\mu(n)$: $n \in \omega\}$ is a pairwise disjoint cover of $\nu$. For each $\alpha \in \nu$, the relation $\Phi_{\sigma,\alpha} \leqslant^* h$ implies the existence of an $n$ with

$\alpha \in \Delta_n$. The pairwise disjoint property follows from the fact that $\Delta$ is monotone nondecreasing.

Claim 2. For each $k$, $\mu$ is subordinate to $\{\sigma_k^{-1}(i)\colon i \in \omega\}$. Claim 2 follows from the inclusions $\mu(n) \subseteq \Delta_n \subseteq \Gamma_{k,n}$.

Claim 3. An infinite number of the terms of $\mu$ are nonempty. Claim 3 follows from subordination.

Sufficiency. Claim. Suppose we have a subordinate partition $\mu$. Then $g = g_{\sigma,\mu}$ is an upper bound for $\{\Phi_{\sigma,\alpha} \colon \alpha \in \nu\}$. Fix $\alpha \in \nu$. Let $f = \Phi_{\sigma,\alpha}$. Let $m \in \omega$ with $\alpha \in \mu(m)$. Then by the definition of $I$, for each $n \geqslant m$, $\sigma_n(\alpha) \in I(n)$. Since $f(n) = \sigma_n(\alpha)$, we have that $g(n)$ is the largest integer in a set of integers that includes $f(n)$. So for each $n \geqslant m$, $f(n) \leqslant g(n)$.

## Reference

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